ADVANCED MODERN ENGINEERING MATHEMATICS

Fifth Edition

Glyn James & Phil Dyke

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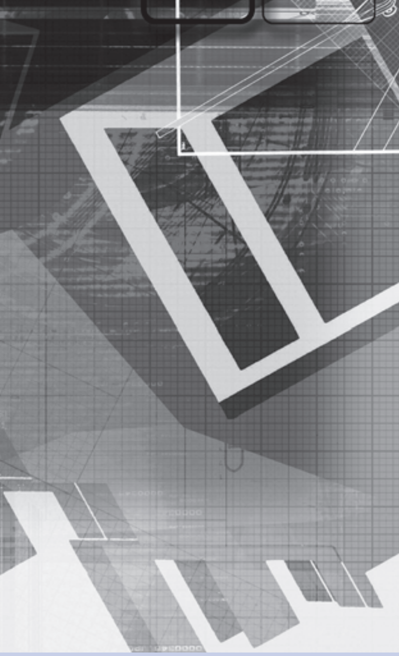

Advanced Modern Engineering Mathematics

Fifth Edition

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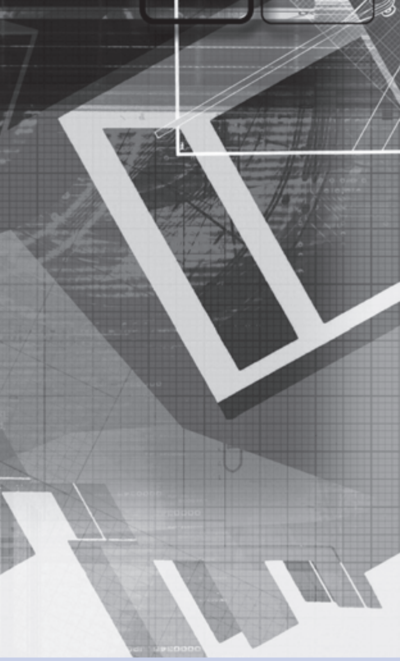

Advanced Modern Engineering Mathematics

Fifth Edition

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Preface

The first edition of this book appeared in 1993, and it could be assumed, wrongly, that its time has passed as 24 years have now elapsed. It is true that all the original authors apart from myself have retired but, in the intervening years the text has been regularly updated and we have now reached the fifth edition. The words of my colleague and predecessor as editor, Professor Glyn James, still ring true. Here is an excerpt from his preface to the fourth edition (2011):

> Throughout the course of history, engineering and mathematics have developed in parallel. All branches of engineering depend on mathematics for their description and there has been a steady flow of ideas and problems from engineering that has stimulated and sometimes initiated branches of mathematics. Thus, it is vital that engineering students receive a thorough grounding in mathematics, with the treatment related to their interests and problems. As with the previous editions, this has been the motivation for the production of this latest edition – a companion text to the fifth edition of *Modern Engineering Mathematics*, this being designed to provide a first-level core studies course in mathematics for undergraduate programmes in all engineering disciplines. Building on the foundations laid in the companion text, this book gives an extensive treatment of some of the more advanced areas of mathematics that have applications in various fields of engineering, particularly as tools for computer-based system modelling, analysis and design. Feedback, from users of the previous editions, on subject content has been highly positive indicating that it is sufficiently broad to provide the necessary second-level, or optional, studies for most engineering programmes, where in each case a selection of the material may be made. Whilst designed primarily for use by engineering students, it is believed that the book is also suitable for use by students of applied mathematics and the physical sciences.
>
> Although the pace of the book is at a somewhat more advanced level than the companion text, the philosophy of learning by doing is retained with continuing emphasis on the development of students' ability to use mathematics with understanding to solve engineering problems. Recognizing the increasing importance of mathematical modelling in engineering practice, many of the worked examples and exercises incorporate mathematical models that are designed both to provide relevance and to reinforce the role of mathematics in various branches of engineering. In addition, each chapter contains specific sections on engineering applications, and these form an ideal framework for individual, or group, study assignments, thereby helping to reinforce the skills of mathematical modelling, which are seen as essential if engineers are to tackle the increasingly complex systems they are being called upon to analyse and design. The importance of numerical methods in problem solving is also recognized, and its treatment is integrated with the analytical work throughout the book.

The position of software use is an important aspect of engineering education. The decision has been taken to use mainly MATLAB but also MAPLE. Students are encouraged to make intelligent use of software and, where appropriate, codes are included, but there is a health warning. The pace of technology shows little signs of lessening, and so in the space of six years, the likely time lapse before a new edition of this text, it is probable that software will continue to be updated, probably annually. There is therefore a real risk that much coding though correct and working at the time of publication could be broken by these updates. Therefore, in this edition the decision has been made not to over-emphasise specific code but to direct students to the companion website or to general principles instead. The software packages, particularly MAPLE, have become easier to use without the need for programming skills. Much is menu driven these days. Here's more from Glyn on the subject that is still true:

> Much of the feedback from users relates to the role and use of software packages, particularly symbolic algebra packages. Without making it an essential requirement the authors have attempted to highlight throughout the text situations where the user could make effective use of software. This also applies to exercises and, indeed, a limited number have been introduced for which the use of such a package is essential. Whilst any appropriate piece of software can be used, the authors recommend the use of MATLAB and/or MAPLE. In this edition reference to the use of these two packages is made throughout the text, with commands or codes introduced and illustrated. When indicated, students are strongly recommended to use these packages to check their solutions to exercises. This is not only to help develop proficiency in their use, but also to enable students to appreciate the necessity of having a sound knowledge of the underpinning mathematics if such packages are to be used effectively. Throughout the book two icons are used:
>
> - An open screen indicates that the use of a software package would be useful (e.g. for checking solutions) but not essential.
> - A closed screen indicates that the use of a software package is essential or highly desirable.

Specific changes in this fifth edition are an improvement in many of the diagrams, taking advantage of present day software, and modernization of the examples and language. Also, the chapter on Applied Probability and Statistics has been significantly modernized by interfacing the presentation with the very powerful software package R. Simply search for 'R Software' and it is a free download. I have been much aided in getting this edition ready for publication by my hardworking colleagues Matthew, Tim and Julian who have joined the editorial team.

Acknowledgements

The authoring team is extremely grateful to all the reviewers and users of the text who have provided valuable comments on previous editions of this book. Most of this has been highly constructive and very much appreciated. The team has continued to enjoy the full support of a very enthusiastic production team at Pearson Education and wishes to thank all those concerned.

Phil Dyke and Glyn James

About the Authors

A new set of authors, Matthew Craven, Tim Reis and Julian Stander under the new editor, one of the original authors, Phil Dyke, have taken on the task of producing this, the fifth edition of *Advanced Modern Engineering Mathematics*.

Phil Dyke is Professor of Applied Mathematics at the University of Plymouth. He was a Head of School for 22 years, 18 of these as Head of Mathematics and Statistics. He has over 45 years teaching and research experience in higher education, much of this teaching engineering students not only mathematics but also marine and coastal engineering. Apart from his contribution to both *Modern Engineering Mathematics* and *Advanced Modern Engineering Mathematics* he is the author of 11 other textbooks ranging in topic from mathematical methods to mechanics and marine physics. He is now semi-retired, but still teaches and writes.

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The original authors are David Burley, Dick Clements, John Searl, Nigel Steele, Jerry Wright together with Phil Dyke. Their short biographies can be found in the previous editions.

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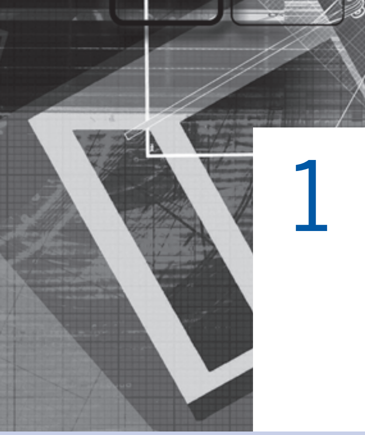

1 Matrix Analysis

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1.1 Introduction

In this chapter we turn our attention again to matrices, first considered in Chapter 5 of *Modern Engineering Mathematics* (MEM), and their applications in engineering. At the outset of the chapter we review the basic results of matrix algebra and briefly introduce vector spaces.

As the reader will be aware, matrices are arrays of real or complex numbers, and have a special, but not exclusive, relationship with systems of linear equations. Such systems occur quite naturally in the process of numerical solution of ordinary differential equations used to model everyday engineering processes. In Chapter 9 we shall see that they also occur in numerical methods for the solution of partial differential equations, for example those modelling the flow of a fluid or the transfer of heat. Systems of linear first-order differential equations with constant coefficients are at the core of the **state-space** representation of linear system models. Identification, analysis and indeed design of such systems can conveniently be performed in the state-space representation, with this form assuming a particular importance in the case of multivariable systems.

In all these areas it is convenient to use a matrix representation for the systems under consideration, since this allows the system model to be manipulated following the rules of matrix algebra. A particularly valuable type of manipulation is **simplification** in some sense. Such a simplification process is an example of a system transformation, carried out by the process of matrix multiplication. At the heart of many transformations are the **eigenvalues** and **eigenvectors** of a square matrix. In addition to providing the means by which simplifying transformations can be deduced, system eigenvalues provide vital information on system stability, fundamental frequencies, speed of decay and long-term system behaviour. For this reason, we devote a substantial amount of space to the process of their calculation, both by hand and by numerical means when necessary. Our treatment of numerical methods is intended to be purely indicative rather than complete, because a comprehensive matrix algebra computational tool kit, such as MATLAB, is now part of the essential armoury of all serious users of mathematics.

In addition to developing the use of matrix algebra techniques, we also demonstrate the techniques and applications of matrix analysis, focusing on the state-space system model widely used in control and systems engineering. Here we encounter the idea of a function of a matrix, in particular the matrix exponential, and we see again the role of the eigenvalues in its calculation. This edition also includes a section on singular value decomposition and the pseudo inverse, together with a brief section on Lyapunov stability of linear systems using quadratic forms.

1.2 Review of matrix algebra

This section contains a summary of the definitions and properties associated with matrices and determinants. A full account can be found in chapters of MEM or elsewhere. It is assumed that readers, prior to embarking on this chapter, have a fairly thorough understanding of the material summarized in this section.

1.2.1 Definitions

(a) An array of real numbers

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

is called an $m \times n$ **matrix** with m rows and n columns. The a_{ij} is referred to as the (ij)th **element** and denotes the element in the ith row and jth column. If $m = n$ then $\boldsymbol{A}$ is called a **square matrix** of order n. If the matrix has one column or one row then it is called a **column vector** or a **row vector** respectively.

(b) In a square matrix $\boldsymbol{A}$ of order n the diagonal containing the elements $a_{11}, a_{22}, \dots, a_{nn}$ is called the **principal** or **leading** diagonal. The sum of the elements in this diagonal is called the **trace** of $\boldsymbol{A}$, that is

$$\text{trace}\,\boldsymbol{A} = \sum_{i=1}^{n} a_{ii}$$

(c) A **diagonal matrix** is a square matrix that has its only non-zero elements along the leading diagonal. A special case of a diagonal matrix is the **unit** or **identity** matrix $\boldsymbol{I}$ for which $a_{11} = a_{22} = \dots = a_{nn} = 1$.

(d) A **zero** or **null** matrix $\mathbf{0}$ is a matrix with every element zero.

(e) The **transposed matrix** $\boldsymbol{A}^{\mathrm{T}}$ is the matrix $\boldsymbol{A}$ with rows and columns interchanged, its i, jth element being a_{ji}.

(f) A square matrix $\boldsymbol{A}$ is called a **symmetric matrix** if $\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}$. It is called **skew symmetric** if $\boldsymbol{A}^{\mathrm{T}} = -\boldsymbol{A}$.

1.2.2 Basic operations on matrices

In what follows the matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are assumed to have the i, jth elements a_{ij}, b_{ij} and c_{ij} respectively.

Equality

The matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are **equal**, that is $\boldsymbol{A} = \boldsymbol{B}$, if they are of the same order $m \times n$ and

$$a_{ij} = b_{ij}, \quad 1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant n$$

Multiplication by a scalar

If λ is a scalar then the matrix $\lambda\boldsymbol{A}$ has elements λa_{ij}.

Addition

We can only add an $m \times n$ matrix $\boldsymbol{A}$ to another $m \times n$ matrix $\boldsymbol{B}$ and the elements of the sum $\boldsymbol{A} + \boldsymbol{B}$ are

$$a_{ij} + b_{ij}, \quad 1 \leqslant i \leqslant m; \quad 1 \leqslant j \leqslant n$$

Properties of addition

(i) commutative law: $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$

(ii) associative law: $(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$

(iii) distributive law: $\lambda(\boldsymbol{A} + \boldsymbol{B}) = \lambda\boldsymbol{A} + \lambda\boldsymbol{B}$, λ scalar

Matrix multiplication

If $\boldsymbol{A}$ is an $m \times p$ matrix and $\boldsymbol{B}$ a $p \times n$ matrix then we define the product $\boldsymbol{C} = \boldsymbol{AB}$ as the $m \times n$ matrix with elements

$$c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}, \quad i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n$$

Properties of multiplication

(i) The commutative law is **not satisfied** in general; that is, in general $\boldsymbol{AB} \neq \boldsymbol{BA}$. Order does matter and we distinguish between $\boldsymbol{AB}$ and $\boldsymbol{BA}$ by the terminology: **pre**-multiplication of $\boldsymbol{B}$ by $\boldsymbol{A}$ to form $\boldsymbol{AB}$ and **post**-multiplication of $\boldsymbol{B}$ by $\boldsymbol{A}$ to form $\boldsymbol{BA}$.

(ii) Associative law: $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$

(iii) If λ is a scalar then

$$(\lambda\boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}(\lambda\boldsymbol{B}) = \lambda\boldsymbol{AB}$$

(iv) Distributive law over addition:

$$(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC} + \boldsymbol{BC} \qquad \boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$$

Note the importance of maintaining order of multiplication as in property (*i*).

(v) If $\boldsymbol{A}$ is an $m \times n$ matrix and if $\boldsymbol{I}_m$ and $\boldsymbol{I}_n$ are the unit matrices of order m and n respectively then

$$\boldsymbol{I}_m\boldsymbol{A} = \boldsymbol{AI}_n = \boldsymbol{A}$$

Properties of the transpose

If $\boldsymbol{A}^{\mathrm{T}}$ is the transposed matrix of $\boldsymbol{A}$ then

(i) $(\boldsymbol{A} + \boldsymbol{B})^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}$

(ii) $(\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{A}$

(iii) $(\boldsymbol{AB})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$

1.2.3 Determinants

The determinant of a square $n \times n$ matrix $\boldsymbol{A}$ is denoted by det $\boldsymbol{A}$ or $|\boldsymbol{A}|$.

If we take a determinant of a matrix and delete row i and column j then the determinant remaining is called the **minor** M_{ij} of the (ij)th element. In general we can take any row i (or column) and evaluate an $n \times n$ determinant $|\boldsymbol{A}|$ as

$$|\boldsymbol{A}| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

A minor multiplied by the appropriate sign is called the **cofactor** A_{ij} of the (ij)th element so $A_{ij} = (-1)^{i+j} M_{ij}$ and thus

$$|\boldsymbol{A}| = \sum_{j=1}^{n} a_{ij} A_{ij}$$

Some useful properties

(i) $|\boldsymbol{A}^{\mathrm{T}}| = |\boldsymbol{A}|$

(ii) $|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$

(iii) A square matrix $\boldsymbol{A}$ is said to be **non-singular** if $|\boldsymbol{A}| \neq 0$ and **singular** if $|\boldsymbol{A}| = 0$.

1.2.4 Adjoint and inverse matrices

Adjoint matrix

The **adjoint** of a square matrix $\boldsymbol{A}$ is the transpose of the matrix of cofactors, so for a 3×3 matrix $\boldsymbol{A}$

$$\text{adj}\,\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^{\mathrm{T}}$$

Properties

(i) $\boldsymbol{A}\,(\text{adj}\,\boldsymbol{A}) = |\boldsymbol{A}|\boldsymbol{I}$

(ii) $|\text{adj}\,\boldsymbol{A}| = |\boldsymbol{A}|^{n-1}$, where n is the order of $\boldsymbol{A}$

(iii) $\text{adj}\,(\boldsymbol{AB}) = (\text{adj}\,\boldsymbol{B})(\text{adj}\,\boldsymbol{A})$

Inverse matrix

Given a square matrix $\boldsymbol{A}$ if we can construct a square matrix $\boldsymbol{B}$ such that

$$\boldsymbol{BA} = \boldsymbol{AB} = \boldsymbol{I}$$

then we call $\boldsymbol{B}$ the inverse of $\boldsymbol{A}$ and write it as $\boldsymbol{A}^{-1}$.

Properties

(i) If $\boldsymbol{A}$ is non-singular then $|\boldsymbol{A}| \neq 0$ and $\boldsymbol{A}^{-1} = (\text{adj } \boldsymbol{A})/|\boldsymbol{A}|$.

(ii) If $\boldsymbol{A}$ is singular then $|\boldsymbol{A}| = 0$ and $\boldsymbol{A}^{-1}$ does not exist.

(iii) $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

All the basic matrix operations may be implemented in MATLAB using simple commands. In MATLAB a matrix is entered as an array, with row elements separated by spaces (or commas) and each row of elements separated by a semicolon(;), or the return key to go to a new line. Thus, for example,

```
A=[1 2 3; 4 0 5; 7 4 2]
```

gives

```
A=
   1 2 3
   4 0 5
   7 4 2
```

Having specified the two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ the operations of addition, subtraction and multiplication are implemented using respectively the commands

```
C=A+B, C=A-B, C=A*B
```

The trace of the matrix $\boldsymbol{A}$ is determined by the command `trace(A)`, and its determinant by `det(A)`.

Multiplication of a matrix $\boldsymbol{A}$ by a scalar is carried out using the command `*`, while raising $\boldsymbol{A}$ to a given power is carried out using the command `^` . Thus, for example, $3\boldsymbol{A}^2$ is determined using the command `C=3*A^2`.

The transpose of a real matrix $\boldsymbol{A}$ is determined using the apostrophe `'` key; that is `C=A'` (to accommodate complex matrices the command `C=A.'` should be used). The inverse of $\boldsymbol{A}$ is determined by `C=inv(A)`.

For matrices involving algebraic quantities, or when exact arithmetic is desirable use of the Symbolic Math Toolbox is required; in which matrices must be expressed in symbolic form using the `sym` command. The command `A=sym(A)` generates the symbolic form of $\boldsymbol{A}$. For example, for the matrix

$$\boldsymbol{A} = \begin{bmatrix} 2.1 & 3.2 & 0.6 \\ 1.2 & 0.5 & 3.3 \\ 5.2 & 1.1 & 0 \end{bmatrix}$$

the commands

```
A=[2.1 3.2 0.6; 1.2 0.5 3.3; 5.2 1.1 0];
A=sym(A)
```

generate

```
A=
  [21/10, 16/5, 3/5]
  [6/5, 1/2, 33/10]
  [26/5, 11/10, 0]
```

Symbolic manipulation can also be undertaken in MATLAB using the MuPAD version of Symbolic Math Toolbox.

Such operations may be performed in Python. Details are not given here, but the interested reader is directed to, for example, *Beginning Python* by Lie Hethand (Springer, 2005). The `numPy` package should be loaded.

1.2.5 Linear equations

In this section we reiterate some definitive statements about the solution of the system of simultaneous linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

or, in matrix notation,

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

that is,

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{1.1}$$

where $\boldsymbol{A}$ is the matrix of coefficients and $\boldsymbol{x}$ is the vector of unknowns. If $\boldsymbol{b} = \boldsymbol{0}$ the equations are called **homogeneous**, while if $\boldsymbol{b} \neq \boldsymbol{0}$ they are called **nonhomogeneous** (or **inhomogeneous**). Considering individual cases:

Case (i): If $\boldsymbol{b} \neq \boldsymbol{0}$ and $|\boldsymbol{A}| \neq 0$ then we have a unique solution $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$.

Case (ii): If $\boldsymbol{b} = \boldsymbol{0}$ and $|\boldsymbol{A}| \neq 0$ we have the trivial solution $\boldsymbol{x} = 0$.

Case (iii): If $\boldsymbol{b} \neq \boldsymbol{0}$ and $|\boldsymbol{A}| = 0$ then we have two possibilities: **either** the equations are inconsistent and we have no solution **or** we have infinitely many solutions.

Case (iv): If $\boldsymbol{b} = \boldsymbol{0}$ and $|\boldsymbol{A}| = 0$ then we have infinitely many solutions.

Case (iv) is one of the most important, since from it we can deduce the important result that **the homogeneous equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has a non-trivial solution if and only if** $|\boldsymbol{A}| = 0$.

Provided that a solution to (1.1) exists it may be determined in MATLAB using the command `x=A\b`. For example, the system of simultaneous equations

$$x + y + z = 6, \quad x + 2y + 3z = 14, \quad x + 4y + 9z = 36$$

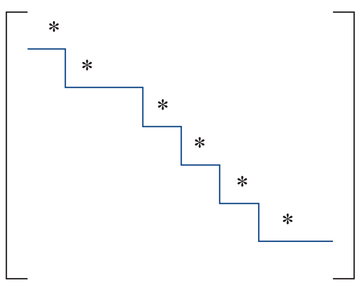
may be written in the matrix form

$$\underset{\boldsymbol{A}}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}} \underset{\boldsymbol{x}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{\boldsymbol{b}}{\begin{bmatrix} 6 \\ 14 \\ 36 \end{bmatrix}}$$

Entering `A` and `b` and using the command `x = A\b` provides the answer $x = 1, y = 2, z = 3$.

1.2.6 Rank of a matrix

We adopt the following constructive definition of the **rank**, rank $\boldsymbol{A}$ of a matrix $\boldsymbol{A}$. First, using elementary row operations, the matrix $\boldsymbol{A}$ is reduced to **echelon form**



in which all the entries below the line are zero, and the leading element, marked *, in each row above the line is non-zero. Then the number of non-zero rows in the echelon form is equal to rank $\boldsymbol{A}$. These are equivalent definitions.

When considering the solution of (1.1) we saw that provided the determinant of the matrix $\boldsymbol{A}$ was not zero we could obtain explicit solutions in terms of the inverse matrix. However, when we looked at cases with zero determinant the results were much less clear. The idea of the rank of a matrix helps to make these results more precise. Defining the **augmented matrix** $(\boldsymbol{A} : \boldsymbol{b})$ for (1.1) as the matrix $\boldsymbol{A}$ with the column $\boldsymbol{b}$ added to it then we can state the results of cases (iii) and (iv) of Section 1.2.5 more clearly as follows:

If $\boldsymbol{A}$ and $(\boldsymbol{A} : \boldsymbol{b})$ have different rank then we have no solution to (1.1). If the two matrices have the same rank then a solution exists, and furthermore the solution will contain $n -$ rank $\boldsymbol{A}$ free parameters.

In MATLAB the rank of the matrix $\boldsymbol{A}$ is generated using the command `rank(A)`. For example, if

$$\boldsymbol{A} = \begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}$$

the commands

```
A=[-1 2 2; 0 0 1; -1 2 0];
rank(A)
```

generate

```
ans=2
```

In MAPLE the command is also `rank(A)`.

1.3 Vector spaces

Vectors and matrices form part of a more extensive formal structure called a **vector space**. The theory of vector spaces underpins many approaches to numerical methods and the approximate solution of many equations that arise in engineering analysis. In this section we shall, briefly, introduce some basic ideas of vector spaces necessary for later work in this chapter.

Definition

A **real vector space** V is a set of objects called **vectors** together with rules for addition and multiplication by real numbers. For any three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ in V and any real numbers α and β the sum $\boldsymbol{a} + \boldsymbol{b}$ and the product $\alpha\boldsymbol{a}$ also belong to V and satisfy the following axioms:

(a) $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$

(b) $\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$

(c) there exists a zero vector $\boldsymbol{0}$ such that

$\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{a}$

(d) for each $\boldsymbol{a}$ in V there is an element $-\boldsymbol{a}$ in V such that

$\boldsymbol{a} + (-\boldsymbol{a}) = \boldsymbol{0}$

(e) $\alpha(\boldsymbol{a} + \boldsymbol{b}) = \alpha\boldsymbol{a} + \alpha\boldsymbol{b}$

(f) $(\alpha + \beta)\boldsymbol{a} = \alpha\boldsymbol{a} + \beta\boldsymbol{a}$

(g) $(\alpha\beta)\boldsymbol{a} = \alpha(\beta\boldsymbol{a})$

(h) $1\boldsymbol{a} = \boldsymbol{a}$